

Spin glass order in spin turbulence

Makoto Tsubota,^{1,2} Yusuke Aoki,¹ and Kazuya Fujimoto¹

¹*Department of Physics, Osaka City University, Sumiyoshi-ku, Osaka 558-8585, Japan*

²*The OCU Advanced Research Institute for Natural Science and Technology (OCARINA), Osaka City University, Sumiyoshi-ku, Osaka 558-8585, Japan*

(Dated: April 18, 2013)

We study numerically the spin turbulence in spin-1 ferromagnetic spinor Bose-Einstein condensates (BECs). Spin turbulence (ST) is characterized by a $-7/3$ power law in the spectrum of the spin-dependent interaction energy. The direction of the spin density vector is spatially disordered but temporally frozen in ST, showing some analogy with the spin glass state. Thus, we introduce the order parameter of spin glass into ST in spinor BECs. When ST develops through some instability, the order parameter grows with a $-7/3$ power law, thus succeeding in describing ST well.

PACS numbers: 67.85.De, 03.75.Lm, 67.25.dk, 47.37.+q

Most flow in nature is actually turbulent [1]. As the velocity increases, the flow generally changes from laminar to turbulent. Turbulence has been a great mystery for a long time mainly because it is a complicated dynamical phenomenon with strong nonlinearity far from an equilibrium state. When we consider such complicated phenomena, it is important to focus on statistical laws. The most important statistical law in turbulence is the Kolmogorov $-5/3$ law. The Kolmogorov energy spectrum refers to $E(k) = C\epsilon^{2/3}k^{-5/3}$, where k is the wave number, ϵ is the energy flux, and C is the Kolmogorov constant of the order unity. The Kolmogorov spectrum is confirmed in fully developed turbulence in classical fluids, being the smoking gun of turbulence.

Apart from these studies in classical fluid dynamics, there have been studies on quantum fluids such as superfluid helium and atomic Bose-Einstein condensates (BECs). These systems are subject to quantum restrictions; the appearance of order parameters means that rotational motion can exist only in the presence of quantized vortices. A quantized vortex is a stable topological defect with quantized circulation. Such vortices give rise to quantum turbulence (QT) [2, 3]. Because quantized vortices are well defined as elements composing a turbulent flow, QT is expected to be an easier system to study than classical turbulence and gives a simpler model of turbulence. Quantum turbulence has long been studied in superfluid helium, whereas atomic BEC has only recently become an important research area.

An important interest on QT is how to characterize the turbulent state. Numerical simulation of the Gross-Pitaevskii (GP) model shows that the spectrum of incompressible kinetic energy obeys the Kolmogorov $-5/3$ law in a uniform system [4, 5] and in a trapped system [6]. Confirmation of such an energy spectrum power law would certainly provide strong proof of turbulence. However, there must be another better way to characterize QT.

Spinor BECs have spin degrees of freedom and exhibit phenomena characteristic of spin [7]. The hydrodynamics

of spinor BECs has recently been studied by several authors [8–11]. In previous studies, we found that hydrodynamic instability in spin-1 spinor BECs exhibits unique behavior owing to their spin degrees of freedom and form spin turbulence (ST) in which the spin density vector has various directions. Although the spin-dependent interaction can be ferromagnetic or antiferromagnetic, we confine ourselves to the case of ferromagnetic interaction in this work. The first study addressed the counterflow between the $m = 1$ and $m = -1$ components in a uniform system, where m is the magnetic quantum number. Through the instability the counterflow leads to ST, in which the spectrum of the spin-dependent interaction energy obeys a $-7/3$ power law [12]; the $-7/3$ power law is a statistical law characteristic of ST and can be understood by the scaling analysis of the time-development equation of the spin density vector. Such a spin-disordered state was experimentally created in a trapped system through the instability of the initial helical structure of spins [13], which allowed us to study ST numerically in a similar situation and confirm the $-7/3$ power law again [14]. An oscillating magnetic field applied to a uniform ferromagnetic system can also create ST with a $-7/3$ power law [15].

These three works [12, 14, 15] reveal the important characteristics of ST. First, the $-7/3$ power law is robust independently of whether the system is uniform or trapped. How the system is excited or how the ST is created has little effect on the power law. Second, observation of the spin motion in all three cases indicates that the spin density vectors become spatially random but temporally frozen as the ST develops and the $-7/3$ power law appears clearly, which reminds us of spin glass. Spin glasses are magnetic systems in which the interactions between the magnetic moments are in conflict with each other [16]. Thus, no conventional long-range order can be established. Nevertheless, these systems exhibit a freezing transition to a state with a kind of order in which the spins are aligned in random directions.

In this paper, we introduce the order parameter of spin

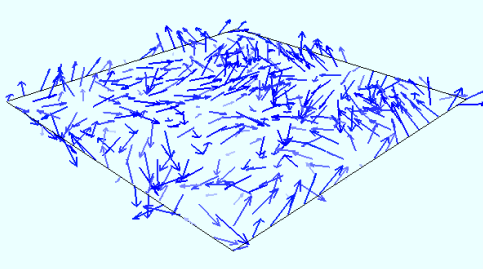


FIG. 1: (Color online) Distribution of spin density vectors in ST at $t/\tau = 4000$ obtained by the counterflow instability with $V_R/c_{\text{sound}} = 0.78$, where V_R , c_{sound} , and τ are the relative velocity of counterflow, the sound velocity, and the characteristic time in a uniform system [12, 21]. The system size is $128\xi \times 128\xi$, where ξ is the coherence length. This numerical calculation starts from the initial state of the counterflow between the $m = 1$ and -1 components, which induces the spin modulation, finally leading to ST. Details of the dynamics are described in [12].

glass [16, 17] to characterize ST in spinor BECs. This introduction is so successful that the order parameter is found to grow with the setup of the $-7/3$ power law. This success paves the way for two innovative approaches. One is to propose a useful order parameter in the turbulence of quantum fluids; such a proposal is impossible in ordinary QT [2]. The other is to connect the study of spinor BECs with that of magnetism including spin glass. In this paper, first we describe how the ST in spinor BECs behaves. As the directions of spins become disordered, the spectrum of the spin-dependent interaction energy starts to exhibit the $-7/3$ power law and the motion of the spins becomes temporally frozen. Second, the introduction of the spin glass order parameter is shown to successfully characterize the ST.

We consider a spin-1 spinor BEC at zero temperature, which is well described by macroscopic wave functions ψ_m ($m = 1, 0, -1$). The wave functions ψ_m obey the GP equation [18, 19]

$$\begin{aligned}
 i\hbar \frac{\partial}{\partial t} \psi_m = & \left(-\frac{\hbar^2}{2M} \nabla^2 + V \right) \psi_m \\
 & + \sum_{n=-1}^1 [-g\mu_B (\mathbf{B} \cdot \hat{\mathbf{S}})_{mn} + q(\mathbf{B} \cdot \hat{\mathbf{S}})_{mn}^2] \psi_n \\
 & + c_0 n \psi_m + c_1 \sum_{n=-1}^1 \mathbf{s} \cdot \hat{\mathbf{S}}_{mn} \psi_n.
 \end{aligned} \quad (1)$$

Here, V and \mathbf{B} are the trapping potential and magnetic field. The parameters M , g , μ_B , and q are the mass of a particle, the Landé g factor, the Bohr magneton, and a coefficient of the quadratic Zeeman effect, respectively. The total density n and the spin density vector \mathbf{s}_i ($i = x, y, z$) are given by $n = \sum_{m=-1}^1 |\psi_m|^2$ and $\mathbf{s}_i = \sum_{m,n=-1}^1 \psi_m^* (\hat{\mathbf{S}}_i)_{mn} \psi_n$ with the spin-1 matrix

$(\hat{\mathbf{S}}_i)_{mn}$. The parameters c_0 and c_1 are the coefficients of the spin-independent and spin-dependent interactions. In our research of ST, we focus on the spin-dependent interaction energy $E_{\text{spin}} = \frac{c_1}{2} \int \mathbf{s}^2 d\mathbf{r}$, whose coefficient c_1 determines whether the system is ferromagnetic ($c_1 < 0$) or antiferromagnetic ($c_1 > 0$). We are interested in the ferromagnetic case where ST clearly exhibits the $-7/3$ power law.

In this paper, we study the two-dimensional ST obtained by (i) the counterflow instability [12], (ii) the instability of spin helical structure [14], and (iii) an oscillating magnetic field [15]. All parameters of the numerical calculation of (i) and (ii) in this paper are the same as in the previous studies. In the case of (iii), we use parameters different from those of the previous study [20]

Figure 1 shows a typical case of ST developing from the instability of the counterflow between the $m = 1$ and $m = -1$ components [12]. As ST develops, the spectrum of the spin-dependent interaction energy begins to exhibit the $-7/3$ power law. Figure 2 shows that the $-7/3$ power law appears in all three cases we have studied: counterflow instability in a uniform system [Fig. 2(a)] [12], instability of the initial helical structure of the spin density vector in a trapped system [Fig. 2(b)] [14], and application of an oscillating magnetic field in a uniform ferromagnetic system [Fig. 2(c)] [15]. Thus, the $-7/3$ power law is universal in ST independently of whether the system is uniform or trapped or how the ST is created. All cases show that the spin density vector is spatially random but temporally frozen; the case of the counterflow instability is shown in the movie [21]. We know such a phenomenon, namely spin glass, which invites us to introduce the order parameter of spin glass.

The order parameter of spin glass is ordinarily introduced as follows: An equilibrium system is supposed to consist of N lattice sites, each site i having spin \mathbf{S}_i ($i = 1, N$). The time average $\langle \mathbf{S}_i(t) \rangle$ and the space average $[\mathbf{S}_i(t)]$ are defined as

$$\langle \mathbf{S}_i(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \mathbf{S}_i(t) dt, \quad [\mathbf{S}_i(t)] = \frac{1}{N} \sum_i \mathbf{S}_i(t). \quad (2)$$

Then it is possible to introduce two order parameters, namely, the magnetization $\mathbf{M} = [\langle \mathbf{S}_i(t) \rangle]$ and $q = [\langle \mathbf{S}_i(t) \rangle^2]$ [16, 17]. If the system is paramagnetic, both \mathbf{M} and q vanish. A ferromagnetic order gives nonzero \mathbf{M} and q . When $\mathbf{M} = 0$ but q is nonzero, the state is called spin glass [17], and the directions of spin are spatially random but temporally frozen.

To apply the order parameter to our present case, we have to consider two things. First, we address the spin density vector $\mathbf{s}(\mathbf{r}, t)$ instead of $\mathbf{S}_i(t)$. Our spinor BECs are usually trapped so that the amplitude of the spin density vector is not uniform. To extract this effect and focus on the direction of the spin density vector, we define the order parameter with the unit vector

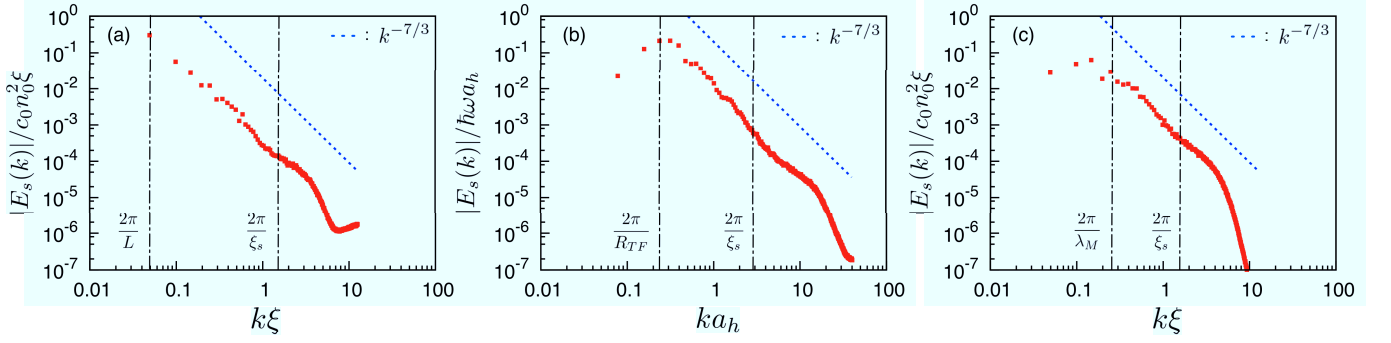


FIG. 2: (Color online) Spectrum of the spin-dependent interaction energy in three cases: (a) counterflow instability in a uniform system, (b) instability of the initial helical structure of the spin density vector in a trapped system, and (c) application of an oscillating magnetic field in a uniform ferromagnetic system. Each spectrum is calculated at (a) $t/\tau = 4000$, (b) $t\omega = 200$, and (c) $t/\tau = 4000$, when ST is fully developed. The parameters ω , a_h , R_{TF} , n_0 , ξ , L , and λ_M are, respectively, the trapping frequency, the harmonic oscillator length, the Thomas-Fermi radius, the total density in the uniform system, the coherence length, the system size, and the wavelength where the energy is injected by the oscillating magnetic field. The details of these parameters except for λ_M are described in our previous paper [12, 14, 15]. The expression for λ_M is obtained by the resonant condition with the oscillating magnetic field [22]. The blue broken shows the $-7/3$ power law. The spectrum in all cases exhibits the $-7/3$ power law in the scaling region [(a) $2\pi/L < k < 2\pi/\xi_s$, (b) $2\pi/R_{TF} < k < 2\pi/\xi_s$, (c) $2\pi/\lambda_M < k < 2\pi/\xi_s$].

$\hat{\mathbf{s}}(\mathbf{r}, t) = \mathbf{s}(\mathbf{r}, t)/|\mathbf{s}(\mathbf{r}, t)|$. The definition of the space average of Eq. (2) is replaced by

$$[\hat{\mathbf{s}}(\mathbf{r}, t)] = \frac{1}{A} \int_A \hat{\mathbf{s}}(\mathbf{r}, t) d\mathbf{r} \quad (3)$$

with system area A . Second, the system starts from some initial state to develop ST, which should be characterized by the time-dependent order parameter. Thus, we introduce the time average of $\hat{\mathbf{s}}(\mathbf{r}, t)$ during the period $[t, t+T]$ as

$$\langle \hat{\mathbf{s}}(\mathbf{r}, t) \rangle_T = \frac{1}{T} \int_t^{t+T} \hat{\mathbf{s}}(\mathbf{r}, t_1) dt_1 \quad (4)$$

and define the time-dependent order parameter

$$q(t) = [\langle \hat{\mathbf{s}}(\mathbf{r}, t) \rangle_T^2]. \quad (5)$$

When we calculate $q(t)$ for our system, we should be careful with how we take T . Generally, a longer T is desirable, but in reality, we should estimate $q(t)$ with some finite T . The criterion for the appropriate value of T would be that T should be longer than some characteristic time of the system. In this system, the velocity characteristic of spin is given by $c_s = \sqrt{c_1 n/M}$ [23]. Then, a typical characteristic time is the system size L divided by the velocity c_s . We will show in the following that this time L/c_s is found to be so long that the system in the initial state becomes sufficiently disturbed by the time. Thus, we take T comparable to L/c_s . In ST, the order parameter $q(t)$ decreases with T , but the dependence is weak. If the spin density vector is completely frozen in ST, $q(t)$ should be unity.

We show how $q(t)$ grows toward ST with a $-7/3$ power law of the spin-dependent interaction energy. Figure 3

shows the time dependence of the power exponent of the spectrum of the spin-dependent interaction energy obtained by the least-squares method and $q(t)$ for (a) counterflow instability in a uniform system, (b) instability of the helical structure in a trapped system, and (c) application of the oscillating magnetic field in a uniform ferromagnetic system. Figures 3(a) and 3(b) show that $q(t)$ increases obviously as the exponent n approaches $-7/3$. Furthermore, the magnetization $|\mathbf{m}(t)| = |[\langle \hat{\mathbf{s}}(\mathbf{r}, t) \rangle_T]|$ is much smaller than $\sqrt{q(t)}$ because $[\mathbf{s}(\mathbf{r}, t)]$ is conserved to keep vanishing under Eq. (1) without the magnetic field. Thus, the two order parameters $q(t)$ and $|\mathbf{m}(t)|$ are found to be effective for describing the ST in the two cases.

In contrast, $q(t)$ of Fig. 3(c) behaves differently from that of Figs. 3(a) and 3(b). In the case of Fig. 3(c), $[\mathbf{s}(\mathbf{r}, t)]$ is not conserved; in Fig. 3(d) we see that $|\mathbf{m}(t)|$ decreases from the initial ferromagnetic state toward ST. In ST, $\sqrt{q(t)}$ of Fig. 3(c) is smaller than that of Figs. 3(a) and 3(b), because the oscillating magnetic field causes fluctuation in the spin density vector and reduces $\sqrt{q(t)}$. However, $\sqrt{q(t)} \sim 0.35$ is much larger than the magnetization $|\mathbf{m}(t)| \sim 0.005$ at $t/\tau = 4000$, which means the growth of the spin glass order in ST. Therefore, the two order parameters $q(t)$ and $|\mathbf{m}(t)|$ are still found to be effective for describing the ST in this case.

Only case (b) is affected by the inhomogeneity. Because the system is trapped, the amplitude of the spin density vector is reduced going from the center to the boundary. The low condensate density near the boundary induces a large fluctuation of the condensate phase, thus causing $\mathbf{s}(\mathbf{r}, t)$ to fluctuate temporally. As a result, the space average of Eq. (3) depends on the area A of the integral. Figure 3(b) shows two cases for which the

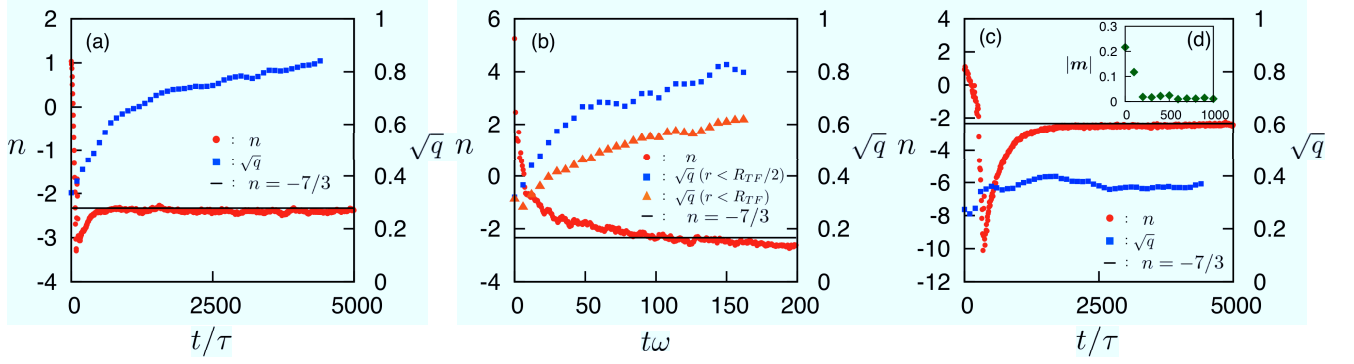


FIG. 3: (Color online) Time development of the exponent n , the spin glass order parameter $\sqrt{q(t)}$, and $|\mathbf{m}(t)|$. n and $\sqrt{q(t)}$ are shown for (a) the counterflow instability in a uniform system, (b) the instability of the initial helical structure of the spin density vector in a trapped system, and (c) the application of the oscillating magnetic field in a uniform ferromagnetic system. The time development of $|\mathbf{m}(t)|$ in the case of (c) is shown in (d). In (b), the squares and triangles show that the calculation of \sqrt{q} is performed in the regions $r < R_{TF}/2$ and $r < R_{TF}$, respectively. The exponents n in (a) \sim (c) are evaluated by the least-squares method in the scaling region defined in the caption of Fig. 2.

radius of A is $R_{TF}/2$ and R_{TF} with the Thomas-Fermi radius being R_{TF} . The larger value of $q(t)$ for $R_{TF}/2$ means that the central spins are more likely to be frozen.

Finally, we note that the time $T = L/c_s$ is so long that the system in the initial state becomes disturbed, which is found by the time development of n . The time T is (a) 572τ , (b) $24/\omega$, and (c) 572τ , respectively. We find that n rapidly changes in the period $0 < t < T$, which means that the distribution of the spin density vector in the wavenumber space changes rapidly too. Thus, in this period, the spin density vector temporally points in various directions. This is also confirmed in the movie of the spin density vector [21]. Therefore, the time L/c_s is appropriate for the calculation of the spin glass order parameter $q(t)$.

In summary, we have studied numerically ST in spin-1 spinor BECs. The spectrum of the spin-dependent interaction energy is found to exhibit a $-7/3$ power law independently of the details of the system or how the ST is created. The direction of the spin density vector is spatially disordered but temporally frozen in ST, which reminds us of the spin glass state. Following spin glass theory, we introduced the order parameter of the spin glass into ST in spinor BECs. The order parameter $q(t)$ of spin glass grows with a $-7/3$ power law, well describing ST. Therefore, ST can be characterized by three quantities: (i) the spin glass order parameter \sqrt{q} , (ii) the magnetization \mathbf{m} , and (iii) the power exponent $n = -7/3$ of the spectrum of the spin-dependent interaction energy. These behaviors should be accessible experimentally.

We thank Shin-ichi Sasa for useful discussions.

-
- [1] P. A. Davidson, *Turbulence: An Introduction for Scientists and Engineers* (Oxford University Press, Oxford, 2004).
 - [2] *Progress in Low Temperature Physics*, edited by W. P. Halperin and M. Tsubota (Elsevier, Amsterdam, 2009), Vol. XVI.
 - [3] M. Tsubota, K. Kasamatsu and M. Kobayashi, in *Novel Superfluids*, edited by K. H. Bennemann and J. B. Ketterson (Oxford University Press, Oxford, 2013), Vol. 1, p. 156.
 - [4] C. Nore, M. Abid, and M. E. Brachet, Phys. Rev. Lett. **78**, 3896 (1997); Phys. Fluids **9**, 2644 (1997).
 - [5] M. Kobayashi and M. Tsubota, Phys. Rev. Lett. **94**, 065302 (2005); J. Phys. Soc. Jpn. **74**, 3248 (2005).
 - [6] M. Kobayashi and M. Tsubota, Phys. Rev. A **76**, 045603 (2007).
 - [7] Y. Kawaguchi and M. Ueda, Phys. Rep. **520**, 253 (2013).
 - [8] A. Lamacraft, Phys. Rev. A **77**, 063622 (2008).
 - [9] R. Barnett, D. Podolsky, and G. Refael, Phys. Rev. B **80**, 024420 (2009).
 - [10] K. Kudo and Y. Kawaguchi, Phys. Rev. A **82**, 053614 (2010); **84**, 043607 (2011).
 - [11] E. Yukawa and M. Ueda, Phys. Rev. A **86**, 063614 (2012).
 - [12] K. Fujimoto and M. Tsubota, Phys. Rev. A **85**, 033642 (2012).
 - [13] M. Vengalattore, S. R. Leslie, J. Guzman, and D. M. Stamper-Kurn, Phys. Rev. Lett. **100**, 170403 (2008).
 - [14] K. Fujimoto and M. Tsubota, Phys. Rev. A **85**, 053641 (2012).
 - [15] Y. Aoki and M. Tsubota, J. Low Temp. Phys. **171**, 382 (2013).
 - [16] K. Binder and A. P. Young, Rev. Mod. Phys. **58**, 801 (1986).
 - [17] D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **35**, 1792 (1975).
 - [18] T. Ohmi and K. Machida, J. Phys. Soc. Jpn. **67**, 1822 (1998).

- [19] T.-L. Ho, Phys. Rev. Lett. **81**, 742 (1998).
- [20] We transform Eq. (1) to the nondimensional equation by using the characteristic length $\hbar/\sqrt{2Mc_0n_0}$ and time \hbar/c_0n_0 . Here, $n_0 = N/L_xL_y$ is the initial total density, where N , L_x , and L_y are the total particle number and the system size in the x and y directions, respectively. The nondimensional system size $\tilde{L}_x \times \tilde{L}_y$ is 128×128 , which is discretized into 512×512 bins. In our numerical calculation, the parameters $\tilde{L}_x\tilde{L}_y$, $c_1\tilde{L}_x\tilde{L}_y/c_0$, $g\mu_B|\mathbf{B}|/c_0n_0$, and $q|\mathbf{B}|/c_0n_0$ are 128^2 , $128^2/20$, $\sqrt{2}/10$, and 0.2, which are the coefficients of the spin-independent interaction, the spin-dependent interaction, the linear Zeeman, and the quadratic Zeeman terms in the nondimensional form of Eq. (1), respectively. We oscillate the magnetic field in the x direction with the frequency $\omega_M = 2\pi c_0n_0/100\hbar$.
- [21] You can view the movie at [\[http://matter.sci.osaka-cu.ac.jp/~bsr/paper_movie/spin_turbulence_movie.html\]](http://matter.sci.osaka-cu.ac.jp/~bsr/paper_movie/spin_turbulence_movie.html)
- [22] The wavelength λ_M is obtained by the resonant condition with the oscillating magnetic field. In the ferromagnetic case, the dispersion relation of the spin wave is a single-particle-like $\hbar^2k^2/2M$ [19]. The resonant condition is $\hbar\omega_M = \hbar^2k_M^2/2M$, which yields $\lambda_M = 2\pi/k_M$.
- [23] The velocity c_s is the spin coherence length $\hbar/\sqrt{|c_1|nM}$ divided by the time $\hbar/|c_1|n$. This time is characteristic of spin because $|c_1|n$ is the order of the spin interaction energy.